

Backward Stochastic Differential Equations with Markov Chains and The Application: Homogenization of PDEs System

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Abstract

Stemmed from the derivation of the optimal control to a stochastic linear-quadratic control problem with Markov jumps, we study one kind of backward stochastic differential equations (BSDEs) that the generator f is affected by a Markovian switching. Then, the case that the Markov chain is involved in a large state space is considered. Following the classical approach, a hierarchical approach is adopted to reduce the complexity and a singularly perturbed Markov chain is involved. We will study the asymptotic property of BSDE with the singularly perturbed Markov chain. At last, as an application of our theoretical result, we show the homogenization of one system of partial differential equations (PDEs) with a singularly perturbed Markov chain.

Keywords: BSDE, Markov chain, weak convergence, homogenization.

1. Introduction

The study of backward stochastic differential equations (BSDEs in short) stemmed from stochastic control problem ([4]) in which a non-linear Ricatti

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BSDE was introduced. Then, after the pioneering work of Pardoux and Peng ([17]) about the general BSDE which was driven by a Brownian motion, BSDEs have been extensively studied in the last twenty years because of their connections with mathematical finance, stochastic control, and partial differential equations (PDEs in short), please refer to [10, 8, 7]. Since then, many researchers devoted their work to more general BSDE, such as BSDE driven by a Lévy process ([1, 15]) and BSDE with respect to both a Brownian motion and a Poisson random measure ([3, 21, 11]). Recently, Cohen and Elliott [5, 6] studied BSDE driven by the martingale part of a Markov chain and its application in finance.

As studied in Zhang and Yin ([25]), we consider the stocks investment models by virtue of hybrid geometric Brownian motion in which both the expected return and volatility depend on a finite state Markov chain. To capture the market trends as well as the various economic factors, a finite state Markov chain α_t , $t \geq 0$, is introduced to represent the general market directions. If our object is to allocate assets into a number of stocks so as to maximize an expected utility within a finite time horizon, it leads to an optimal control problem. By virtue of maximum principle method, we need to introduce an adjoint equation to deal with this optimization problem. The adjoint equation and the state equation form forward backward stochastic differential equations (FBSDEs in short) system. Here the adjoint equation will be a kind of BSDE with a Markov chain.

Motivated by such adjoint equation, in our paper, we consider the following BSDE with a Markov chain:

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s, \alpha_s) ds - \int_t^T Z_s dB_s,$$

where $\alpha = \{\alpha_t; 0 \leq t \leq T\}$ is a continuous-time Markov chain independent of the Brownian motion B . It is noted that this BSDE is different from the one studied in Cohen and Elliott ([5]), and it can be considered as that its generator is disturbed by random environment and takes a set of discrete values which can be described by a Markov chain.

When study the solvability for BSDE with a Markov chain, the classic method with the contraction mapping cannot be directly used for the lack of suitable filtration and corresponding Itô's representation theorem. In this paper, inspired by the method dealing with the BSDE with doubly Brownian motion ([19]), we construct a new filtration and give the corresponding extended Itô's representation theorem.

When various factors are considered, the underlying Markov chain inevitably has a large state space, and the corresponding BSDE becomes increasingly complicated. It is rationale that the change rates of states display a two-time-scale behavior, a fast-time scale and a slow varying one. Under this case, a small parameter $\varepsilon > 0$ can be introduced and the singularly perturbed Markov chain is involved.

In this paper, we consider the case that the states of the underlying Markov chain are divided into a number of weakly irreducible classes such that the Markov chain fluctuates rapidly among different states, and jumps less frequently among those classes. To reduce the complexity, a small parameter ($\varepsilon > 0$) is introduced to reflect the different rate of changes among different states. As shown in Zhang and Yin ([23]), it leads to a singularly perturbed Markovian models with two-time scale, the actual time t and the stretched time $\frac{t}{\varepsilon}$. By aggregating the states in each irreducible class into a single, a limit aggregated Markov chain with a considerably smaller space can be obtained and its asymptotic probability distribution is studied. Such asymptotic theory has many applications in optimal control problem and mathematical finance. We refer interested readers to [25, 24].

In this paper, we will focus on the asymptotic property of BSDE with a singularly perturbed Markov chain. Following the averaged approach that aggregating the states according to their jump rates, we will show that the distribution of $(Y_t, \int_t^T Z_s d\bar{B}_s)$ can be seen as an asymptotic distribution to $(Y_t^\varepsilon, \int_t^T Z_s^\varepsilon dB_s)$, where $(Y^\varepsilon, Z^\varepsilon)$ and (Y, Z) satisfy:

$$Y_t^\varepsilon = \xi + \int_t^T f(s, Y_s^\varepsilon, \alpha_s^\varepsilon) ds - \int_t^T Z_s^\varepsilon dB_s$$

and

$$Y_t = \xi + \int_t^T \bar{f}(s, Y_s, \bar{\alpha}_s) ds - \int_t^T Z_s d\bar{B}_s.$$

Here $\bar{\alpha}$ and $\bar{f}(s, Y_s, \bar{\alpha}_s)$ are respectively the limit aggregated Markov chain and the averaged generator with respect to the quasi stationary distributions of the singularly perturbed Markov chain. Compared to the original BSDE with the singularly perturbed Markov chain, the limit BSDE depends on a Markov chain with a much smaller state space. Thus the complexity is reduced.

It is well known that BSDEs provide a probabilistic representation for the solution of a large class of quasi-linear second order partial differential

equations (PDEs in short) ([3, 18, 19, 20, 13]). Thus BSDEs provide a probabilistic tool to study the homogenization of PDEs, which is the process of replacing rapidly varying coefficients by new ones thus the solutions are close. In this paper, as an application of our theoretical result, after showing the relation between our BSDE and one system of semi-linear PDE, we will show the homogenization result of one system of semi-linear PDE with a singularly perturbed Markov chain.

This paper is organized as following. In section 2, we study the solvability of BSDE with a Markov chain. Section 3 is devoted to the case that Markov chain has a large space. Under Jakubowski S-topology ([9]) which is weaker than Skorohod's topology, we present the asymptotic property of BSDE with a singularly perturbed Markov chain. In section 4, we show the application of our theoretical results in the homogenization of one system of semi-linear PDE with a singularly perturbed Markov chain. For the terseness of the main text of our paper, we put part of technical proofs for some results in Appendix.

2. BSDEs with Markov Chains

Let (Ω, \mathcal{F}, P) be a probability space and $T > 0$ be fixed. $\{\mathcal{H}_t, 0 \leq t \leq T\}$ is a filtration on the space satisfying the usual condition. $B = \{B_t; 0 \leq t \leq T\}$ with $B_0 = 0$ is a d -dimensional \mathcal{H}_t -Brownian motion, and $\alpha = \{\alpha_t; 0 \leq t \leq T\}$ is a continuous-time Markov chain independent of B with the state space $\mathcal{M} = \{1, 2, \dots, m\}$. Suppose the generator of the Markov chain $Q = (q_{ij})_{m \times m}$ is given by

$$P\{\alpha(t + \Delta) = j | \alpha(t) = i\} = \begin{cases} q_{ij}\Delta + o(\Delta), & \text{if } i \neq j \\ 1 + q_{ii}\Delta + o(\Delta), & \text{if } i = j \end{cases}$$

where $\Delta > 0$. Here $q_{ij} \geq 0$ is the transition rate from i to j if $i \neq j$, while $q_{ii} = -\sum_{j=1, i \neq j}^m q_{ij}$.

Throughout this paper, we introduce the following notations: $|\cdot|$ is the norm in the corresponding space; A' is the transpose of matrix A ; $L^p(\mathcal{H}_t; R^n)$ is the space of R^n -valued \mathcal{H}_t -adapted random variable ξ satisfying $E(|\xi|^p) < \infty$; $M_{\mathcal{H}_t}^2(0, T; R^n)$ denotes the space of R^n -valued \mathcal{H}_t -adapted stochastic processes $\varphi = \{\varphi_t; t \in [0, T]\}$ satisfying $E \int_0^T |\varphi_t|^2 dt < \infty$; $S_{\mathcal{H}_t}^2(0, T; R^n)$ is the space of R^n -valued \mathcal{H}_t -adapted continuous stochastic processes $\varphi = \{\varphi_t; t \in [0, T]\}$ satisfying $E(\sup_{0 \leq t \leq T} |\varphi_t|^2) < \infty$.

2.1. Motivation

To study the stochastic optimal control problem with a Markov chain, we introduce an adjoint equation, then the state equation and adjoint equation form a kind of FBSDEs with a Markov chain. Here the adjoint equation will be a BSDE with Markov chain. We give the following linear quadratic (LQ in short) optimal control problem as an example.

Consider the following stochastic LQ control problem with Markov jumps

$$\min. J(v) = \frac{1}{2} E \left(\int_0^T ((x_t^v)' R(t, \alpha_t) x_t^v + v_t' N(t, \alpha_t) v_t) dt + (x_T^v)' Q(\alpha_T) x_T^v \right) \quad (1a)$$

$$\text{s. t. } \begin{cases} dx_t^v = (A(t, \alpha_t) x_t^v + B(t, \alpha_t) v_t) dt + (C(t, \alpha_t) x_t^v + D(t, \alpha_t) v_t) dB_t \\ x_0^v = a \in R^n \end{cases} \quad (1b)$$

where $A(t, \alpha_t) = A_i(t), B(t, \alpha_t) = B_i(t), C(t, \alpha_t) = C_i(t), D(t, \alpha_t) = D_i(t), R(t, \alpha_t) = R_i(t), N(t, \alpha_t) = N_i(t)$ when $\alpha_t = i$ ($i = 1, \dots, m$), and they are uniformly bounded \mathcal{F}_t^B -adapted processes with appropriate dimensions. $Q(\alpha_T) = Q_i$ when $\alpha_T = i$ ($i = 1, \dots, m$), and it is nonnegative symmetric matrices-valued \mathcal{F}_T^B -measurable random variable. Besides, $R_i(t)$ is nonnegative symmetric matrices-valued, $N_i(t)$ is positive symmetric matrices-valued and the inverse $N_i(t)^{-1}$ is bounded. The set of all \mathcal{H}_t -adapted admissible controls is $\mathcal{U}_{ad} \equiv M_{\mathcal{H}_t}^2(0, T; R^{n_u \times d})$, and our aim is to find an admissible control u such that $J(u) = \inf_{v \in \mathcal{U}_{ad}} J(v)$.

There are many literatures on this kind of LQ optimal control problem with Markov jumps (1) and its application, such as [26, 12, 24] and their references. Different to their methods that constructing the optimal control via the solution of Riccati equation, we will use the FBSDEs approach.

Theorem 2.1. *If the following FBSDE admits a unique solution (x_t, y_t, z_t)*

$$\begin{cases} dx_t = (A(t, \alpha_t) x_t + B(t, \alpha_t) (-N^{-1}(t, \alpha_t) (B'(t, \alpha_t) y_t + D'(t, \alpha_t) z_t))) dt \\ \quad + (C(t, \alpha_t) x_t + D(t, \alpha_t) (-N^{-1}(t, \alpha_t) (B'(t, \alpha_t) y_t + D'(t, \alpha_t) z_t))) dB_t, \\ -dy_t = (A'(t, \alpha_t) y_t + C'(t, \alpha_t) z_t + R(t, \alpha_t) x_t) dt - z_t dB_t, \\ x_0 = a, \quad y_T = Q(\alpha_T) x_T. \end{cases} \quad (2)$$

Then

$$u_t = -N^{-1}(t, \alpha_t) (B'(t, \alpha_t)y_t + D'(t, \alpha_t)z_t), 0 \leq t \leq T$$

is the unique optimal control for the LQ problem (1)

Proof. Firstly, we will prove that $\{u = u_t; 0 \leq t \leq T\}$ is an optimal control for the LQ problem (1).

From the forward equation of (2), we can see that x is the corresponding system state trajectory of u . For an arbitrary admissible control v , denote x^v as the corresponding system state trajectory, then

$$\begin{aligned} & J(v) - J(u) \\ &= \frac{1}{2}E \left(\int_0^T ((x_t^v - x_t)'R(t, \alpha_t)(x_t^v - x_t) + (v_t - u_t)'N(t, \alpha_t)(v_t - u_t) \right. \\ &\quad \left. + 2x_t'R(t, \alpha_t)(x_t^v - x_t) + 2u_t'N(t, \alpha_t)(v_t - u_t))dt \right. \\ &\quad \left. + (x_T^v - x_T)'Q(\alpha_T)(x_T^v - x_T) + 2x_T'Q(\alpha_T)(x_T^v - x_T) \right). \end{aligned}$$

Applying Itô's formula to $y_t'(x_t^v - x_t)$, we have

$$\begin{aligned} & Ex_T'Q(\alpha_T)(x_T^v - x_T) \\ &= E \int_0^T ((y_t'B(t, \alpha_t) + z_t'D(t, \alpha_t))(v_t + N^{-1}(t, \alpha_t)(B'(t, \alpha_t)y_t + D'(t, \alpha_t)z_t)) \\ &\quad - x_t'R(t, \alpha_t)(x_t^v - x_t))dt \\ &= E \int_0^T ((y_t'B(t, \alpha_t) + z_t'D(t, \alpha_t))(v_t - u_t) - x_t'R(t, \alpha_t)(x_t^v - x_t))dt. \end{aligned}$$

As R, Q are nonnegative and N is positive, we have

$$\begin{aligned} J(v) - J(u) &\geq E \int_0^T (y_t'B(t, \alpha_t) + z_t'D(t, \alpha_t) + u_t'N(t, \alpha_t))(v_t - u_t)dt \\ &= 0. \end{aligned}$$

So $u(t) = -N^{-1}(t, \alpha_t) (B'(t, \alpha_t)y_t + D'(t, \alpha_t)z_t)$ is an optimal control.

Uniqueness: Assume that u^1 and u^2 are both optimal controls with $J(u^1) = J(u^2) = \gamma \geq 0$, and the corresponding trajectories are x^1 and x^2 . Due to the linear property of the system, the trajectories corresponding to $\frac{u^1 + u^2}{2}$ is $\frac{x^1 + x^2}{2}$.

From the classical parallelogram rule, R , Q are nonnegative, and N is positive, there exists $\delta > 0$ such that

$$\begin{aligned}
& 2\gamma \\
&= J(u^1) + J(u^2) \\
&= 2J\left(\frac{u^1 + u^2}{2}\right) + E\left(\left(\frac{x_T^1 - x_T^2}{2}\right)' Q(\alpha_T) \left(\frac{x_T^1 - x_T^2}{2}\right) + \int_0^T \left(\left(\frac{x_t^1 - x_t^2}{2}\right)' \right. \right. \\
&\quad \left. \left. R(t, \alpha_t) \left(\frac{x_t^1 - x_t^2}{2}\right) + \left(\frac{u_t^1 - u_t^2}{2}\right)' N(t, \alpha_t) \left(\frac{u_t^1 - u_t^2}{2}\right)\right) dt\right) \\
&\geq 2J\left(\frac{u^1 + u^2}{2}\right) + E \int_0^T \left(\frac{u_t^1 - u_t^2}{2}\right)' N(t, \alpha_t) \left(\frac{u_t^1 - u_t^2}{2}\right) dt \\
&\geq 2\gamma + \frac{\delta}{4} E \int_0^T |u_t^1 - u_t^2|^2 dt
\end{aligned}$$

Thus $E \int_0^T |u_t^1 - u_t^2|^2 dt \leq 0$ which yields that $u^1 = u^2$. \square

2.2. BSDEs with Markov chains

The derivation of the optimal control for the above LQ problem (1) can be regarded as one motivation for us to study BSDEs with Markov jumps. In this subsection, we will study the solvability to the following BSDE with a Markov chain firstly:

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s, \alpha_s) ds - \int_t^T Z_s dB_s. \quad (3)$$

Let \mathcal{N} denote the class of all P -null sets of \mathcal{F} . For each $t \in [0, T]$, we define $\mathcal{F}_t = \mathcal{F}_t^B \vee \mathcal{F}_{t,T}^\alpha \vee \mathcal{N}$ where for any process $\{\eta_t; 0 \leq t \leq T\}$, $\mathcal{F}_{t,T}^\eta = \sigma\{\eta_r; t \leq r \leq T\}$ and $\mathcal{F}_t^\eta = \mathcal{F}_{0,t}^\eta$. For convenience, we denote $M^2(0, T; R^n) = M_{\mathcal{F}_t}^2(0, T; R^n)$ and $S^2(0, T; R^n) = S_{\mathcal{F}_t}^2(0, T; R^n)$.

Thereinafter, we make the following assumption:

Assumption 2.1. (i) $\xi \in L^2(\mathcal{F}_T; R^k)$; (ii) $f : \Omega \times [0, T] \times R^k \times R^{k \times d} \times \mathcal{M} \rightarrow R^k$ satisfies that $\forall (y, z) \in R^k \times R^{k \times d}$, $\forall i \in \mathcal{M}$, $f(\cdot, y, z, i) \in M_{\mathcal{F}_t}^2(0, T; R^k)$, and $\exists \mu > 0$, such that $\forall i \in \mathcal{M}$, $\forall (\omega, t) \in \Omega \times [0, T]$, $(y_1, z_1), (y_2, z_2) \in R^k \times R^{k \times d}$,

$$|f(t, y_1, z_1, i) - f(t, y_2, z_2, i)| \leq \mu(|y_1 - y_2| + |z_1 - z_2|).$$

Our main result in this section is in the following theorem.

Theorem 2.2. *Under Assumption 2.1, there exists a unique solution pair $(Y, Z) \in S^2(0, T; R^k) \times M^2(0, T; R^{k \times d})$ for BSDE (3).*

The proof of Theorem 2.2 consists of three steps.

Step 1: Extension of Itô's representation theorem

It is noted that $\{\mathcal{F}_t; 0 \leq t \leq T\}$ is neither increasing nor decreasing, and it does not constitute a filtration. Inspired by the method handling the BSDE with doubly Brown motions ([19]), we define a filtration $(\mathcal{G}_t)_{0 \leq t \leq T}$ by

$$\mathcal{G}_t \triangleq \mathcal{F}_t^B \vee \mathcal{F}_T^\alpha \vee \mathcal{N}$$

For the filtration $(\mathcal{G}_t)_{0 \leq t \leq T}$, we give the following extension of Itô's representation theorem. This result and its corollary play key roles during the proof of Theorem 2.2.

Proposition 2.1. *For $N \in L^2(\mathcal{G}_T; R^k)$, there exist a unique random variable $N_0 \in L^2(\mathcal{F}_T^\alpha; R^k)$ and a unique stochastic process $Z = \{Z_t; 0 \leq t \leq T\} \in M_{\mathcal{G}_t}^2(0, T; R^{k \times d})$ such that*

$$N = N_0 + \int_0^T Z_t dB_t, \quad 0 \leq t \leq T. \quad (4)$$

Actually, $N_0 = E(N|\mathcal{F}_T^\alpha)$.

During the derivation of Proposition 2.1, we need the following two lemmas.

Lemma 2.1. ([22]) *If $X, Y: \Omega \rightarrow R^d$ are two given functions, Y is $\sigma(X)$ -measurable if and only if there exists a Borel measurable function $g: R^d \rightarrow R^d$ such that $Y = g(X)$.*

Lemma 2.2. (Doob's martingale convergence theorem) *Let $\{\mathcal{F}_t; t \geq 0\}$ be a filtration on the space (Ω, \mathcal{F}, P) , $X \in L^1(\mathcal{F}; R^d)$, then*

$$E(X|\mathcal{F}_t) \rightarrow E(X|\mathcal{F}_\infty), \quad \text{as } t \rightarrow \infty, \text{ a.s. and in } L^1 \text{ as well.}$$

Proof. Existence: Let $\{t_i\}_{i \geq 0}, \{t'_j\}_{j \geq 0}$ be two dense subsets of $[0, T]$ where $t_0 = t'_0 = 0$. For each integer $n, m \geq 0$, let $\mathcal{G}_{n,m}$ be the σ -algebra generated by $\alpha_{t_0}, \alpha_{t_1}, \dots, \alpha_{t_n}, B_{t'_0}, B_{t'_1}, \dots, B_{t'_m}$, i.e., $\mathcal{G}_{n,m} = \sigma\{\alpha_{t_0}, \alpha_{t_1}, \dots, \alpha_{t_n}, B_{t'_0}, B_{t'_1},$

$\dots, B_{t'_m}\}$. Obviously, $\mathcal{G}_{n,m} \subset \mathcal{G}_{n+1,m}$, $\mathcal{G}_{n,m} \subset \mathcal{G}_{n,m+1}$, $\mathcal{G}_{n,m} \subset \mathcal{G}_{n+1,m+1}$, $\sigma(\cup_{m=1}^{\infty} \mathcal{G}_{n,m}) = \mathcal{F}_T^B \vee \sigma\{\alpha_{t_0}, \alpha_{t_1}, \dots, \alpha_{t_n}\}$, and $\sigma(\cup_{n,m=0}^{\infty} \mathcal{G}_{n,m}) = \mathcal{G}_T$.

For $N \in L^2(\mathcal{G}_T; R^k)$, $\forall n, m$, by Lemma 2.1 and Lemma 2.2, there exists a Borel measurable function $N_{n,m} : \mathcal{M}^{n+1} \times R^{(m+1) \times d} \rightarrow R^k$ such that

$$E(N|\mathcal{G}_{n,m}) = N_{n,m}(\alpha_{t_0}, \alpha_{t_1}, \dots, \alpha_{t_n}, B_{t'_0}, B_{t'_1}, \dots, B_{t'_m})$$

$$N_{n,m}(\alpha_{t_0}, \alpha_{t_1}, \dots, \alpha_{t_n}, B_{t'_0}, B_{t'_1}, \dots, B_{t'_m}) \rightarrow E(N|\mathcal{F}_T^B \vee \sigma\{\alpha_{t_0}, \alpha_{t_1}, \dots, \alpha_{t_n}\})$$

Denote $N_n(\alpha_{t_0}, \alpha_{t_1}, \dots, \alpha_{t_n}) \triangleq E(N|\mathcal{F}_T^B \vee \sigma\{\alpha_{t_0}, \alpha_{t_1}, \dots, \alpha_{t_n}\})$, it can be rewritten as

$$N_n(\alpha_{t_0}, \alpha_{t_1}, \dots, \alpha_{t_n}) = \sum_{i_0, i_1, \dots, i_n=1}^m I_{\{(\alpha_{t_0}, \alpha_{t_1}, \dots, \alpha_{t_n}) = (i_0, i_1, \dots, i_n)\}} N_n(i_0, i_1, \dots, i_n)$$

where $N_n(i_0, i_1, \dots, i_n)$ is \mathcal{F}_T^B -measurable.

For $(i_0, i_1, \dots, i_n) \in \mathcal{M}^{n+1}$, applying Itô's representation theorem,

$$N_n(i_0, i_1, \dots, i_n) = N_0(i_0, i_1, \dots, i_n) + \int_0^T Z_t(i_0, i_1, \dots, i_n) dB_t$$

where $N_0(i_0, i_1, \dots, i_n)$ is a constant and $Z(i_0, i_1, \dots, i_n) \in M_{\mathcal{F}_t^B}^2(0, T; R^{k \times d})$.

Denote

$$N_0(\alpha_{t_0}, \alpha_{t_1}, \dots, \alpha_{t_n}) = \sum_{i_0, i_1, \dots, i_n=1}^m I_{\{(\alpha_{t_0}, \alpha_{t_1}, \dots, \alpha_{t_n}) = (i_0, i_1, \dots, i_n)\}} N_0(i_0, i_1, \dots, i_n),$$

$$\text{and } Z_t(\alpha_{t_0}, \alpha_{t_1}, \dots, \alpha_{t_n}) = \sum_{i_0, i_1, \dots, i_n=1}^m I_{\{(\alpha_{t_0}, \alpha_{t_1}, \dots, \alpha_{t_n}) = (i_0, i_1, \dots, i_n)\}} Z_t(i_0, i_1, \dots, i_n).$$

Clearly, $N_0(\alpha_{t_0}, \alpha_{t_1}, \dots, \alpha_{t_n}) \in L^2(\mathcal{F}_T^B; R^k)$, $\{Z_t(\alpha_{t_0}, \alpha_{t_1}, \dots, \alpha_{t_n}); 0 \leq t \leq T\} \in M_{\mathcal{G}_t}^2(0, T; R^{k \times d})$, and we have

$$N_n(\alpha_{t_0}, \alpha_{t_1}, \dots, \alpha_{t_n}) = N_0(\alpha_{t_0}, \alpha_{t_1}, \dots, \alpha_{t_n}) + \int_0^T Z_t(\alpha_{t_0}, \alpha_{t_1}, \dots, \alpha_{t_n}) dB_t. \quad (5)$$

In the remained part, we will prove that as $n \rightarrow \infty$, both side of (5) are Cauchy sequences.

For the left hand side, as $n \rightarrow \infty$, with the definition of $N_n(\alpha_{t_0}, \alpha_{t_1}, \dots, \alpha_{t_n})$ and Lemma 2.2, we have

$$N_n(\alpha_{t_0}, \alpha_{t_1}, \dots, \alpha_{t_n}) \rightarrow N.$$

For the right hand side, since

$$\begin{aligned}
& E \left(\int_0^T Z(\alpha_{t_0}, \alpha_{t_1}, \dots, \alpha_{t_n}) dB_t | \mathcal{F}_T^\alpha \right) \\
&= \sum_{i_0, i_1, \dots, i_n=1}^m I_{\{(\alpha_{t_0}, \alpha_{t_1}, \dots, \alpha_{t_n}) = (i_0, i_1, \dots, i_n)\}} E \left(\int_0^T Z(i_0, i_1, \dots, i_n) dB_t | \mathcal{F}_T^\alpha \right) \\
&= 0.
\end{aligned} \tag{6}$$

Thus $N_0(\alpha_{t_0}, \alpha_{t_1}, \dots, \alpha_{t_n}) = E(N_n(\alpha_{t_0}, \alpha_{t_1}, \dots, \alpha_{t_n}) | \mathcal{F}_T^\alpha)$. As $n \rightarrow \infty$, we can conclude that

$$N_0(\alpha_{t_0}, \alpha_{t_1}, \dots, \alpha_{t_n}) \rightarrow E(N | \mathcal{F}_T^\alpha).$$

Now let us consider the sequence $\{Z_t(\alpha_{t_0}, \alpha_{t_1}, \dots, \alpha_{t_n}); 0 \leq t \leq T\}$. For $n, m \geq 0$,

$$\begin{aligned}
& E \int_0^T |Z_t(\alpha_{t_0}, \alpha_{t_1}, \dots, \alpha_{t_n}) - Z_t(\alpha_{t_0}, \alpha_{t_1}, \dots, \alpha_{t_m})|^2 dt \\
&= E \left(\int_0^T (Z_t(\alpha_{t_0}, \alpha_{t_1}, \dots, \alpha_{t_n}) - Z_t(\alpha_{t_0}, \alpha_{t_1}, \dots, \alpha_{t_m})) dB_t \right)^2 \\
&= E \left(E(N | \mathcal{F}_T^B \vee \sigma\{\alpha_{t_0}, \alpha_{t_1}, \dots, \alpha_{t_n}\}) - E(N | \mathcal{F}_T^B \vee \sigma\{\alpha_{t_0}, \alpha_{t_1}, \dots, \alpha_{t_m}\}) \right. \\
&\quad \left. - N_0(\alpha_{t_0}, \alpha_{t_1}, \dots, \alpha_{t_n}) + N_0(\alpha_{t_0}, \alpha_{t_1}, \dots, \alpha_{t_m}) \right)^2 \\
&\leq 2E(E(N | \mathcal{F}_T^B \vee \sigma\{\alpha_{t_0}, \alpha_{t_1}, \dots, \alpha_{t_n}\})^2 - E(N | \mathcal{F}_T^B \vee \sigma\{\alpha_{t_0}, \alpha_{t_1}, \dots, \alpha_{t_m}\}))^2 \\
&\quad + 2E(N_0(\alpha_{t_0}, \alpha_{t_1}, \dots, \alpha_{t_n}) - N_0(\alpha_{t_0}, \alpha_{t_1}, \dots, \alpha_{t_m}))^2 \\
&\rightarrow 0, \quad \text{as } n, m \rightarrow \infty.
\end{aligned}$$

Thus $\{Z_t(\alpha_{t_0}, \alpha_{t_1}, \dots, \alpha_{t_n}); 0 \leq t \leq T\}$ is a Cauchy sequence in $M_{\mathcal{G}_t}^2(0, T; R^{k \times d})$. Hence it converges to some $Z \in M_{\mathcal{G}_t}^2(0, T; R^{k \times d})$.

Denote $N_0 = E(N | \mathcal{F}_T^\alpha)$, we can conclude that (5) converges to the extended Itô's representation (4).

Uniqueness: By virtue of equation (6) and the fact that as $n \rightarrow \infty$, $\{Z_t(\alpha_{t_0}, \alpha_{t_1}, \dots, \alpha_{t_n}); 0 \leq t \leq T\}$ is a Cauchy sequence, we have $E(\int_0^T Z_t dB_t | \mathcal{F}_T^\alpha) = 0$. Then, for $(N_0, Z), (N'_0, Z')$ satisfying the extended Itô's representation (4), we get $N_0 = N'_0$ by taking conditional expectation with respect to \mathcal{F}_T^α . Uniqueness of Z follows easily from the fact that

$$E \int_0^T |Z_t - Z'_t|^2 dt = E \left(\int_0^T (Z_t - Z'_t) dB_t \right)^2 = E(N_0 - N'_0)^2 = 0.$$

□

The following corollary is useful in the proof of Theorem 2.2 and its proof is similar to Proposition 2.1.

Corollary 2.1. *For $t \leq T$, we consider the filtration $(\mathcal{N}_s)_{t \leq s \leq T}$ defined by $\mathcal{N}_s = \mathcal{F}_s^B \vee \mathcal{F}_{t,T}^\alpha$. For $N \in L^2(\mathcal{N}_T; R^k)$, there exists a unique stochastic process $Z = \{Z_s; t \leq s \leq T\} \in M_{\mathcal{N}_s}^2(t, T; R^{k \times d})$ such that*

$$N = E(N|\mathcal{N}_t) + \int_t^T Z_s dB_s.$$

Step 2: The special case: the generator f is independent of y and z .

Proposition 2.2. *Under Assumption 2.1, the following BSDE*

$$Y_t = \xi + \int_t^T f(s, \alpha_s) ds - \int_t^T Z_s dB_s, \quad 0 \leq t \leq T \quad (7)$$

has a solution pair $(Y, Z) \in S^2(0, T; R^k) \times M^2(0, T; R^{k \times d})$.

Proof. From Assumption 2.1 and Hölder inequality, we obtain

$$E \left(\int_0^T f(s, \alpha_s) ds \right)^2 \leq CE \int_0^T |f(s, \alpha_s)|^2 ds \leq C \sum_{i=1}^m E \int_0^T |f(s, i)|^2 ds < \infty$$

which yields

$$\xi + \int_0^T f(s, \alpha_s) ds \in L^2(\mathcal{G}_T; R^k)$$

For the filtration $(\mathcal{G}_t)_{0 \leq t \leq T}$ where $\mathcal{G}_t = \mathcal{F}_t^B \vee \mathcal{F}_T^\alpha \vee \mathcal{N} = \mathcal{F}_t^B \vee \mathcal{F}_{t,T}^\alpha \vee \mathcal{F}_t^\alpha \vee \mathcal{N} = \mathcal{F}_t \vee \mathcal{F}_t^\alpha$, we can define the following \mathcal{G}_t -measurable square integrable martingale

$$N_t = E \left(\xi + \int_0^T f(s, \alpha_s) ds | \mathcal{G}_t \right), \quad 0 \leq t \leq T.$$

By the extended Itô's representation theorem (Proposition 2.1), there exist $N_0 \in L^2(\mathcal{F}_T^\alpha; R^k)$ and $Z = \{Z_t; 0 \leq t \leq T\} \in M_{\mathcal{G}_t}^2(0, T; R^{k \times d})$ such that

$$N_t = N_0 + \int_0^t Z_s dB_s, \quad 0 \leq t \leq T.$$

For $t \in [0, T]$, we define

$$Y_t = N_t - \int_0^t f(s, \alpha_s) ds, \quad \text{i.e., } Y_t = E \left(\xi + \int_t^T f(s, \alpha_s) ds | \mathcal{G}_t \right). \quad (8)$$

It is easy to verify that the \mathcal{G}_t -measurable process (Y, Z) satisfies BSDE (7) and $Y \in M_{\mathcal{G}_t}^2(0, T; R^k)$. We refer interested reader to [17, 16] for the detailed verification.

The left work is to show that the processes $Y = \{Y_t; 0 \leq t \leq T\}$ and $Z = \{Z_t; 0 \leq t \leq T\}$ are \mathcal{F}_t -measurable, i.e. $\mathcal{F}_t^B \vee \mathcal{F}_{t,T}^\alpha$ -measurable. $\forall t \in [0, T]$, we denote $\vartheta = \xi + \int_t^T f(s, \alpha_s) ds$, ϑ is $\mathcal{F}_T^B \vee \mathcal{F}_{t,T}^\alpha$ -measurable.

Let $\{\bar{t}_i\}_{i \geq 0}, \{\bar{t}_j\}_{j \geq 0}$ be respectively dense subsets of $[t, T]$ and $[0, T]$, with $\bar{t}_0 = t$ and $\bar{t}'_0 = 0$. For each integer $n, m \geq 0$, let $\bar{\mathcal{G}}_{n,m}$ be the σ -algebra generated by $\alpha_{\bar{t}_0}, \alpha_{\bar{t}_1}, \dots, \alpha_{\bar{t}_n}, B_{\bar{t}'_0}, B_{\bar{t}'_1}, \dots, B_{\bar{t}'_m}$, i.e., $\bar{\mathcal{G}}_{n,m} = \sigma\{\alpha_{\bar{t}_0}, \alpha_{\bar{t}_1}, \dots, \alpha_{\bar{t}_n}, B_{\bar{t}'_0}, B_{\bar{t}'_1}, \dots, B_{\bar{t}'_m}\}$. Obviously, $\bar{\mathcal{G}}_{n,m} \subset \bar{\mathcal{G}}_{n+1,m+1}$, $\bar{\mathcal{G}}_{n,m} \subset \bar{\mathcal{G}}_{n+1,m}$, $\bar{\mathcal{G}}_{n,m} \subset \bar{\mathcal{G}}_{n,m+1}$, and $\sigma(\cup_{n,m=0}^\infty \bar{\mathcal{G}}_{n,m}) = \mathcal{F}_T^B \vee \mathcal{F}_{t,T}^\alpha$.

From Lemma 2.1, for each n, m , there exists a Borel measurable function $\vartheta_{nm} : \mathcal{M}^{n+1} \times R^{(m+1) \times d} \rightarrow R^k$ such that

$$E[\vartheta | \bar{\mathcal{G}}_{n,m}] = \vartheta_{nm}(\alpha_{\bar{t}_0}, \alpha_{\bar{t}_1}, \dots, \alpha_{\bar{t}_n}, B_{\bar{t}'_0}, B_{\bar{t}'_1}, \dots, B_{\bar{t}'_m}).$$

Since $I_{\{(\alpha_{\bar{t}_0}, \alpha_{\bar{t}_1}, \dots, \alpha_{\bar{t}_n}) = (i_0, i_1, \dots, i_n)\}} \in \mathcal{F}_{t,T}^\alpha \subset \mathcal{F}_T^\alpha$, we have

$$\begin{aligned} & E(\vartheta_{nm}(\alpha_{\bar{t}_0}, \alpha_{\bar{t}_1}, \dots, \alpha_{\bar{t}_n}, B_{\bar{t}'_0}, B_{\bar{t}'_1}, \dots, B_{\bar{t}'_m}) | \mathcal{F}_t^B \vee \mathcal{F}_T^\alpha) \\ &= E \left(\sum_{i_0, i_1, \dots, i_n=1}^m I_{\{(\alpha_{\bar{t}_0}, \alpha_{\bar{t}_1}, \dots, \alpha_{\bar{t}_n}) = (i_0, i_1, \dots, i_n)\}} \right. \\ & \quad \left. \vartheta_{nm}(i_0, i_1, \dots, i_n, B_{\bar{t}'_0}, B_{\bar{t}'_1}, \dots, B_{\bar{t}'_m}) | \mathcal{F}_t^B \vee \mathcal{F}_T^\alpha \right) \\ &= \sum_{i_0, i_1, \dots, i_n=1}^m I_{\{(\alpha_{\bar{t}_0}, \alpha_{\bar{t}_1}, \dots, \alpha_{\bar{t}_n}) = (i_0, i_1, \dots, i_n)\}} \\ & \quad E(\vartheta_{nm}(i_0, i_1, \dots, i_n, B_{\bar{t}'_0}, B_{\bar{t}'_1}, \dots, B_{\bar{t}'_m}) | \mathcal{F}_t^B \vee \mathcal{F}_T^\alpha). \end{aligned}$$

For the reason that $\vartheta_{nm}(i_0, i_1, \dots, i_n, B_{\bar{t}'_0}, B_{\bar{t}'_1}, \dots, B_{\bar{t}'_m})$ is \mathcal{F}_T^B -measurable, by Itô's representation theorem, we know that there exist $\nu_0^{n,m}(i_0, i_1, \dots, i_n)$

$\in R^k$ and $Z^{n,m}(i_0, i_1, \dots, i_n) \in L^2_{\mathcal{F}_t^B}(0, T; R^{k \times d})$ such that

$$\begin{aligned} & E(\vartheta_{nm}(i_0, i_1, \dots, i_n, B_{\bar{t}_0}, B_{\bar{t}_1}, \dots, B_{\bar{t}_m}) | \mathcal{F}_t^B \vee \mathcal{F}_T^\alpha) \\ &= E\left(\nu_0^{n,m}(i_0, i_1, \dots, i_n) + \int_0^T Z_r^{n,m}(i_0, i_1, \dots, i_n) dB_r \middle| \mathcal{F}_t^B \vee \mathcal{F}_T^\alpha\right) \\ &= \nu_0^{n,m}(i_0, i_1, \dots, i_n) + \int_0^t Z_r^{n,m}(i_0, i_1, \dots, i_n) dB_r \end{aligned}$$

is \mathcal{F}_t^B -measurable. Therefore

$$\begin{aligned} & E(\vartheta_{nm}(\alpha_{\bar{t}_0}, \alpha_{\bar{t}_1}, \dots, \alpha_{\bar{t}_n}, B_{\bar{t}_0}, B_{\bar{t}_1}, \dots, B_{\bar{t}_m}) | \mathcal{F}_t^B \vee \mathcal{F}_T^\alpha) \\ &= \sum_{i_0, i_1, \dots, i_n=1}^m I_{\{(\alpha_{\bar{t}_0}, \alpha_{\bar{t}_1}, \dots, \alpha_{\bar{t}_n}) = (i_0, i_1, \dots, i_n)\}} \\ & \quad E(\vartheta_{nm}(i_0, i_1, \dots, i_n, B_{\bar{t}_0}, B_{\bar{t}_1}, \dots, B_{\bar{t}_m}) | \mathcal{F}_t^B \vee \mathcal{F}_T^\alpha) \end{aligned}$$

is $\mathcal{F}_t^B \vee \mathcal{F}_{t,T}^\alpha$ -measurable. With Lemma 2.2, as $n, m \rightarrow \infty$,

$$E[\vartheta | \bar{\mathcal{G}}_{n,m}] \rightarrow E[\vartheta | \mathcal{F}_T^B \vee \mathcal{F}_{t,T}^\alpha] = \vartheta.$$

Thus $E[\vartheta | \mathcal{F}_t^B \vee \mathcal{F}_T^\alpha]$, i.e., Y_t , is also $\mathcal{F}_t^B \vee \mathcal{F}_{t,T}^\alpha$ -measurable.

Considering

$$\int_t^T Z_s dB_s = -Y_t + \xi + \int_t^T f(s, \alpha_s) ds,$$

its right side is $\mathcal{F}_T^B \vee \mathcal{F}_{t,T}^\alpha$ -measurable. With Corollary 2.1, we know $\forall t < s$, Z_s is $\mathcal{F}_s^B \vee \mathcal{F}_{t,T}^\alpha$ -measurable. Then, by the continuous property of the Markov chain α , we obtain that Z_s is $\mathcal{F}_s^B \vee \mathcal{F}_{s,T}^\alpha$ -measurable.

Together with the Burkholder-Davis-Gundy inequality and the form of BSDE (7), we can conclude that $\{Y_t; 0 \leq t \leq T\}$ is continuous and satisfies $E(\sup_{0 \leq t \leq T} |Y_t|^2) < \infty$. It yields that $Y \in S^2(0, T; R^k)$. \square

Step 3: The general case: Proof of Theorem 2.2.

Proof. Firstly, we define a mapping I from $M^2(0, T; R^k \times R^{k \times d})$ into itself such that $(Y, Z) \in S^2(0, T; R^k) \times M^2(0, T; R^{k \times d})$ is the solution to BSDE (3) iff it is a fixed point of I .

For a constant $\beta > 0$, we introduce the following equivalent norm of $M^2(0, T; R^k \times R^{k \times d})$

$$\|v(\cdot)\|_\beta = \left(E \int_0^T |v_s|^2 e^{\beta s} ds \right)^{\frac{1}{2}}.$$

For $(y, z) \in M^2(0, T; R^k \times R^{k \times d})$, we set

$$Y_t = \xi + \int_t^T f(s, y_s, z_s, \alpha_s) ds - \int_t^T Z_s dB_s.$$

From Assumption 2.1 and Hölder's inequality,

$$\begin{aligned} & E \left(\int_0^T f(s, y_s, z_s, \alpha_s) ds \right)^2 \\ & \leq 2E \left(\int_0^T (f(s, y_s, z_s, \alpha_s) - f(s, 0, 0, \alpha_s)) ds \right)^2 + 2E \left(\int_0^T f(s, 0, 0, \alpha_s) ds \right)^2 \\ & \leq C \left(E \int_0^T (|y_s|^2 + |z_s|^2) ds + \sum_{i=1}^m E \int_0^T |f(s, 0, 0, i)|^2 ds \right) < \infty \end{aligned}$$

which yields that

$$\xi + \int_0^T f(s, y_s, z_s, \alpha_s) ds \in L^2(\mathcal{G}_T; R^k).$$

From Proposition 2.2, we can define the following contraction mapping under the norm $\|\cdot\|_\beta$

$$I((y, z)) = (Y, Z) : M^2(0, T; R^k \times R^{k \times d}) \rightarrow M^2(0, T; R^k \times R^{k \times d}).$$

The proof of contraction property is similar to [17, 10, 16]. For the compactness of the paper, the detail is omit here.

Together with the form of BSDE (3) and Burkholder-Davis-Gundy inequality, $Y \in S^2(0, T; R^k)$. Thus, by the fixed point theorem, we know that BSDE (3) has a unique solution pair. \square

3. BSDEs with Singularly Perturbed Markov Chains

In this section, after recalling several relevant results of singularly perturbed Markov chains given by Zhang and Yin ([23]), we will consider the asymptotic property of BSDE with a singularly perturbed Markov chain. Following the averaging approach to aggregate the states according to their jump rates and replace the actual coefficient with its average with respect to the quasi stationary distributions of the singularly perturbed Markov chain, we get the asymptotic probability distribution of the solution to the BSDE with an limit averaged Markov chain which has a much smaller state space than the original one.

3.1. Relevant results of singularly perturbed Markov chains

Focused on a continuous-time ε -dependent singularly perturbed Markov chain $\alpha^\varepsilon = \{\alpha_t^\varepsilon; 0 \leq t \leq T\}$ which have the generator $Q^\varepsilon = \frac{1}{\varepsilon}\tilde{Q} + \hat{Q}$, where \tilde{Q} and \hat{Q} are time-invariant generators, with $\tilde{Q} = \text{diag}(\tilde{Q}^1, \dots, \tilde{Q}^l)$. The state space can be decomposed as $\mathcal{M} = \{1, 2, \dots, m\} = \mathcal{M}_1 \cup \dots \cup \mathcal{M}_l$, $\mathcal{M}_k = \{s_{k1}, \dots, s_{km_k}\}$, and for $k \in \{1, \dots, l\}$, \tilde{Q}^k is the weakly irreducible generator³ corresponding to the states in \mathcal{M}_k . The generator \tilde{Q} dictates the fast motion of the Markov chain and \hat{Q} governs the slow motion, i.e., the underlying Markov chain fluctuates rapidly in a single group \mathcal{M}_k and jumps less frequently among groups \mathcal{M}_k and \mathcal{M}_j for $k \neq j$.

As shown in [23], when the states in \mathcal{M}_k are lumped into a single state, all such states are coupled by \hat{Q} . By defining $\bar{\alpha}_t^\varepsilon = k$, when $\alpha_t^\varepsilon \in \mathcal{M}_k$, we can obtain the aggregated process $\bar{\alpha}^\varepsilon = \{\bar{\alpha}_t^\varepsilon; 0 \leq t \leq T\}$ containing l states. The process $\bar{\alpha}^\varepsilon$ is not necessarily Markovian, but it converges weakly to a continuous-time Markov chain $\bar{\alpha}$.

Proposition 3.1. ([23]) (i) $\bar{\alpha}^\varepsilon$ converges weakly to $\bar{\alpha}$ generated by

$$\bar{Q} = \text{diag}(\nu^1, \dots, \nu^l) \hat{Q} \text{diag}(\mathbb{I}_{m_1}, \dots, \mathbb{I}_{m_l})$$

as $\varepsilon \rightarrow 0$, where ν^k is the quasi-stationary distribution of \tilde{Q}^k , $k = 1, \dots, l$, and $\mathbb{I}_k = (1, \dots, 1)' \in R^k$.

³A generator Q is called weakly irreducible if the system of equations $\nu Q = 0$ and $\sum_{i=1}^m \nu_i = 1$ has a unique nonnegative solution. This nonnegative solution $\nu = (\nu_1, \dots, \nu_m)$ is called the quasi-stationary distribution of Q .

(ii) For any bounded deterministic function $\beta(\cdot)$,

$$E \left(\int_s^T (I_{\{\alpha_t^\varepsilon = s_{kj}\}} - \nu_j^k I_{\{\bar{\alpha}_t^\varepsilon = k\}}) \beta(t) dt \right)^2 = O(\varepsilon), \forall k = 1, \dots, l, \forall j = 1, \dots, m_k.$$

Here I_A is the indicator function of a set A .

3.2. Weak convergence of BSDEs with singularly perturbed Markov chains

In this subsection, denote $D(0, T; R^k)$ as the Skorohod space of càdlàg trajectories endowed with the Jakubowski S-topology ([9]) which is weaker than the Skorohod topology. As shown in the appendix of [2], the tightness criteria under this S-topology is the same as the “Meyer-Zheng tightness criteria” used in [14].

Here, we only consider the asymptotic property of the solution to the following BSDE with a singularly perturbed Markov chain where the generator f does not depend on Z^ε ,

$$Y_t^\varepsilon = \xi + \int_t^T f(s, Y_s^\varepsilon, \alpha_s^\varepsilon) ds - \int_t^T Z_s^\varepsilon dB_s, \quad (9)$$

For the difficulty to study the general case that the generator f depends on Z^ε , we refer interested reader to the explanation in section 6 of [16].

Firstly, we make the following assumption:

Assumption 3.1. (i) $\xi \in L^2(\mathcal{F}_T^B; R^k)$. (ii) For $f : [0, T] \times R^k \times \mathcal{M} \rightarrow R^k$, there exists a constant $C > 0$ such that $\sup_{\substack{0 \leq t \leq T \\ 1 \leq i \leq m}} |f(t, 0, i)| \leq C$.

Theorem 3.1. Under Assumption 2.1 and Assumption 3.1, the sequence of process $(Y_t^\varepsilon, \int_0^t Z_s^\varepsilon dB_s)$ converges in distribution to the process $(Y_t, \int_0^t Z_s d\bar{B}_s)$ as $\varepsilon \rightarrow 0$, when probability measures on $D(0, T; R^{2k})$ equipped with the Jakubowski S-topology. Here (Y, Z) is the solution pair to the following BSDE with the limit averaged Markov chain

$$Y_t = \xi + \int_t^T \bar{f}(s, Y_s, \bar{\alpha}_s) ds - \int_t^T Z_s d\bar{B}_s, \quad (10)$$

$\bar{B} = \{\bar{B}_t; 0 \leq t \leq T\}$ with $\bar{B}_0 = 0$ is a d -dimensional Brownian motion, $\bar{\alpha}$ is defined in subsection 3.1, and $\bar{f}(s, y, i) = \sum_{j=1}^{m_i} \nu_j^i f(t, y, s_{ij})$ for $i \in \bar{\mathcal{M}} = \{1, \dots, l\}$.

Remark 3.1. *It is obvious that the limit BSDE depends on the limit averaged Markov chain $\bar{\alpha}$ with a state space much smaller than that of the original singularly perturbed Markov chain α^ε . Moreover, as $\varepsilon \rightarrow 0$, the $\mathcal{F}_T^{\alpha^\varepsilon}$ -measurable random variables sequence (Y_0^ε) converges in distribution to the random variable Y_0 which is $\mathcal{F}_T^{\bar{\alpha}}$ -measurable.*

For the proof of Theorem 3.1, we follow a classical approach as in [20, 2] to prove the weak convergence of BSDE: after showing the tightness and convergence for $(Y_t^\varepsilon, \int_0^t Z_s^\varepsilon dB_s)$, we identify the limit.

Step 1: Tightness and convergence for $(Y_t^\varepsilon, \int_0^t Z_s^\varepsilon dB_s)$.

Proposition 3.2. *Under Assumption 2.1 and Assumption 3.1, BSDE (9) and BSDE (10) have unique solutions $(Y^\varepsilon, Z^\varepsilon)$ and $(Y, Z) \in S^2(0, T; R^k) \times M^2(0, T; R^{k \times d})$. Moreover, there exists a positive constant C such that $\forall \varepsilon > 0$,*

$$\begin{aligned} E \left(\sup_{0 \leq t \leq T} |Y_t^\varepsilon|^2 + \int_0^T (Z_t^\varepsilon)^2 dt \right) &\leq C, \\ E \left(\sup_{0 \leq t \leq T} |Y_t|^2 + \int_0^T (Z_t)^2 dt \right) &\leq C. \end{aligned}$$

Proof. For BSDE (9), by Theorem 2.2, the existence and uniqueness of solution $(Y^\varepsilon, Z^\varepsilon)$ is obtained for all $\varepsilon > 0$.

Using Itô's formula to $|Y_s^\varepsilon|^2$ on $[t, T]$, we get the following from Schwartz's inequality,

$$\begin{aligned} &|Y_t^\varepsilon|^2 + \int_t^T |Z_s^\varepsilon|^2 ds \\ &= |\xi|^2 + 2 \int_t^T Y_s^\varepsilon f(s, Y_s^\varepsilon, Z_s^\varepsilon, \alpha_s) ds - 2 \int_t^T Y_s^\varepsilon Z_s^\varepsilon dB_s \\ &\leq |\xi|^2 + 2 \int_t^T ((1 + \mu^2)|Y_s^\varepsilon|^2 + |f(s, 0, 0, \alpha_s^\varepsilon)|^2) ds - 2 \int_t^T Y_s^\varepsilon Z_s^\varepsilon dB_s \end{aligned}$$

here μ is the Lipschitz constant of f which is independent of ε . By taking expectation, we can deduce

$$E \left(|Y_t^\varepsilon|^2 + \frac{1}{2} \int_t^T |Z_s^\varepsilon|^2 ds \right) \leq |\xi|^2 + 2 \int_t^T ((1 + \mu^2)|Y_s^\varepsilon|^2 + |f(s, 0, 0, \alpha_s^\varepsilon)|^2) ds.$$

From Gronwall's lemma, we get

$$E \left(|Y_t^\varepsilon|^2 + \int_t^T |Z_s^\varepsilon|^2 ds \right) \leq CE \left(|\xi|^2 + \int_0^T |f(s, 0, 0, \alpha_s^\varepsilon)|^2 ds \right) \leq C,$$

and then the estimation for $(Y^\varepsilon, Z^\varepsilon)$ is obtained from the Burkholder-Davis-Gundy inequality.

From the form of \bar{f} presented in Theorem 3.1, we know that \bar{f} also satisfies Assumption 2.1 and Assumption 3.1, thus the estimation about (Y, Z) can be obtained similarly. \square

We set $M_t^\varepsilon = \int_0^t Z_s^\varepsilon dB_s$ for the convenience. Thus BSDE (9) can be rewritten as

$$Y_t^\varepsilon = \xi + \int_t^T f(s, Y_s^\varepsilon, \alpha_s^\varepsilon) ds - (M_T^\varepsilon - M_t^\varepsilon). \quad (11)$$

Proposition 3.3. *The sequence of $(Y^\varepsilon, M^\varepsilon)$ is tight on the space $D(0, T; R^k) \times D(0, T; R^k)$.*

Proof. Let $\mathcal{G}_t^\varepsilon = \mathcal{F}_t^B \vee \mathcal{F}_T^{\alpha^\varepsilon} \vee \mathcal{N}$, we define the conditional variation

$$CV(Y^\varepsilon) = \sup E \left(\sum_i |E(Y_{t_{i+1}}^\varepsilon - Y_{t_i}^\varepsilon | \mathcal{G}_{t_i}^\varepsilon)| \right)$$

where the supreme is taken over all partitions of the interval $[0, T]$.

From the Proof of Proposition 2.2, we know that M^ε is a $\mathcal{G}_t^\varepsilon$ -martingale. It follows that

$$CV(Y^\varepsilon) \leq E \int_0^T |f(s, Y_s^\varepsilon, \alpha_s^\varepsilon)| ds.$$

From (ii) of Assumption 2.1, (ii) of Assumption 3.1, and Proposition 3.2, we know

$$\sup_\varepsilon \left(CV(Y^\varepsilon) + \sup_{0 \leq t \leq T} E|Y_t^\varepsilon| + \sup_{0 \leq t \leq T} E|M_t^\varepsilon| \right) < \infty.$$

Thus the ‘‘Meyer-Zheng tightness criteria’’ ([2, 14]) is fully satisfied and the result is followed. \square

Together with the properties of Y^ε obtained above, the following proposition can be seen as an obvious result of Lemma 7.3 in [25].

Proposition 3.4. Suppose $g(t, x)$ is a function defined on $[0, T] \times R^m$ satisfying that $g(\cdot, \cdot)$ is Lipschitz continuous with x and $\forall x \in R^m$, either $|g(t, x)| \leq K(1 + |x|)$ or $|g(t, x)| \leq K$. Denote $\pi_{ij}^\varepsilon(t) = \pi_{ij}^\varepsilon(t, \alpha_t^\varepsilon)$, with $\pi_{ij}^\varepsilon(t, \alpha) = I_{\{\alpha = s_{ij}\}} - \nu_j^i I_{\{\alpha \in M_i\}}$, then for any $k = 1, \dots, l, j = 1, \dots, m_k$,

$$\sup_{0 < t \leq T} E \left| \int_0^t g(s, Y_s^\varepsilon) \pi_{ij}^\varepsilon(s, \alpha_s^\varepsilon) ds \right| \rightarrow 0, \quad \text{as } \varepsilon \rightarrow 0.$$

Step 2: Identification of the limit.

From Proposition 3.3, we know that there exists a subsequence of $(Y^\varepsilon, M^\varepsilon)$, which we still denote by $(Y^\varepsilon, M^\varepsilon)$, and which converges in distribution on the space $D(0, T; R^k) \times D(0, T; R^k)$ toward a càdlàg process (\bar{Y}, \bar{M}) . Furthermore, there exists a countable subset D of $[0, T]$, such that $(Y^\varepsilon, M^\varepsilon)$ converges in finite-distribution to (\bar{Y}, \bar{M}) on D^c .

Proposition 3.5. For the limit process (\bar{Y}, \bar{M}) , we have

(i) For every $t \in [0, T] - D$,

$$\bar{Y}_t = \xi + \int_t^T \bar{f}(s, \bar{Y}_s, \bar{\alpha}_s) ds - (\bar{M}_T - \bar{M}_t).$$

(ii) For a d -dimensional Brownian motion $\bar{B} = \{\bar{B}_t; 0 \leq t \leq T\}$ with $\bar{B}_0 = 0$, \bar{Y} is measurable with $\mathcal{H}_t = \mathcal{F}_t^{\bar{B}} \vee \mathcal{F}_t^{\bar{\alpha}}$, then \bar{M} is a \mathcal{H}_t -martingale.

Proof. From Proposition 3.4, as $\varepsilon \rightarrow 0$,

$$\sup_{0 \leq t \leq T} E \left| \int_0^t f(s, Y_s^\varepsilon, s_{ij}) (I_{\{\alpha_s^\varepsilon = s_{ij}\}} - \nu_j^i I_{\{\alpha_s^\varepsilon \in M_i\}}) ds \right| \rightarrow 0.$$

Since $(Y^\varepsilon, \bar{\alpha}^\varepsilon)$ converge weakly to $(\bar{Y}, \bar{\alpha})$,

$$\int_0^t \bar{f}(s, Y_s^\varepsilon, \bar{\alpha}_s^\varepsilon) ds \text{ converges in distribution to } \int_0^t \bar{f}(s, \bar{Y}_s, \bar{\alpha}_s) ds \text{ on } C(0, T; R^k).$$

Thus

$$\begin{aligned} & \int_0^t f(s, Y_s^\varepsilon, \alpha_s^\varepsilon) ds \\ &= \int_0^t \sum_{i=1}^l \sum_{j=1}^{m_i} f(s, Y_s^\varepsilon, s_{ij}) I_{\{\alpha_s^\varepsilon = s_{ij}\}} \\ &= \int_0^t \sum_{i=1}^l \sum_{j=1}^{m_i} f(s, Y_s^\varepsilon, s_{ij}) (I_{\{\alpha_s^\varepsilon = s_{ij}\}} - \nu_j^i I_{\{\alpha_s^\varepsilon \in M_i\}}) ds + \int_0^t \bar{f}(s, Y_s^\varepsilon, \bar{\alpha}_s^\varepsilon) ds. \end{aligned}$$

As $\varepsilon \rightarrow 0$, passing to the limit in the backward component of the BSDE (11), we can derive assertion (i).

Now, we prove assertion (ii).

For any $0 \leq t_1 \leq t_2 \leq T$, Φ_{t_1} is a continuous mapping from $C(0, t_1; R^d) \times D(0, t_1; R^k) \times D(0, T; \mathcal{M})$. $\forall \varepsilon > 0$, since M^ε is a martingale with respect to $\mathcal{G}_t^\varepsilon = \mathcal{F}_T^{\alpha^\varepsilon} \vee \mathcal{F}_t^B$, Y^ε and $\bar{\alpha}^\varepsilon$ are $\mathcal{G}_t^\varepsilon$ -adapted, we know

$$E \left(\Phi_{t_1}(B, Y^\varepsilon, \bar{\alpha}^\varepsilon) \left(Y_{t_2}^\varepsilon - Y_{t_1}^\varepsilon + \int_{t_1}^{t_2} f(s, Y_s^\varepsilon, \bar{\alpha}_s^\varepsilon) ds \right) \right) = 0$$

and

$$E \left(\Phi_{t_1}(B, Y^\varepsilon, \bar{\alpha}^\varepsilon) \int_0^\delta (M_{t_2+r}^\varepsilon - M_{t_1+r}^\varepsilon) dr \right) = 0,$$

here B is the Brownian motion.

From the weak convergence of $(Y^\varepsilon, \bar{\alpha}^\varepsilon)$ to $(\bar{Y}, \bar{\alpha})$, $\int_0^t \bar{f}(s, Y_s^\varepsilon, \bar{\alpha}_s^\varepsilon) ds$ converges in distribution to $\int_0^t \bar{f}(s, \bar{Y}_s, \bar{\alpha}_s) ds$ on $C(0, T; R^k)$. For a d -dimensional Brownian motion $\bar{B} = \{\bar{B}_t; 0 \leq t \leq T\}$ with $\bar{B}_0 = 0$, from the fact that \bar{B} has the same probability distribution with B and $E(\sup_{0 \leq t \leq T} |M_t^\varepsilon|^2) \leq C$, we obtain

$$E \left(\Phi_{t_1}(\bar{B}, \bar{Y}, \bar{\alpha}) \left(\bar{Y}_{t_2} - \bar{Y}_{t_1} + \int_{t_1}^{t_2} \bar{f}(s, \bar{Y}_s, \bar{\alpha}_s) ds \right) \right) = 0$$

and

$$E \left(\Phi_{t_1}(\bar{B}, \bar{Y}, \bar{\alpha}) \int_0^\delta (\bar{M}_{t_2+r} - \bar{M}_{t_1+r}) dr \right) = 0.$$

Dividing the second identity by δ , letting $\delta \rightarrow 0$, and exploiting the right continuity, we obtain that

$$E(\Phi_{t_1}(\bar{B}, \bar{Y}, \bar{\alpha})(\bar{M}_{t_2} - \bar{M}_{t_1})) = 0.$$

From the freedom choice of t_1 , t_2 , and Φ_{t_1} , we deduce that \bar{M} is a \mathcal{H}_t -martingale. \square

Proposition 3.6. *Let $\{(Y_t, Z_t); 0 \leq t \leq T\}$ be the unique solution of BSDE (10), then $\forall t \in [0, T]$,*

$$E|Y_t - \bar{Y}_t|^2 + E \left([\bar{M} - \int_0^\cdot Z_r d\bar{B}_r]_T - [\bar{M} - \int_0^\cdot Z_r d\bar{B}_r]_t \right) = 0.$$

Proof. Let $M_t = \int_0^t Z_r d\bar{B}_r$, by the proof of Proposition 2.2, we know that M_t is a $\mathcal{F}_t^{\bar{B}} \vee \mathcal{F}_T^{\bar{\alpha}}$ -martingale.

From Itô's formula and Proposition 3.5, we know that

$$\begin{aligned} & E|Y_t - \bar{Y}_t|^2 + E([M - \bar{M}]_T - [M - \bar{M}]_t) \\ &= 2E \int_t^T (\bar{f}(s, Y_s, \bar{\alpha}_s) - \bar{f}(s, \bar{Y}_s, \bar{\alpha}_s)) (Y_s - \bar{Y}_s) ds \\ &\leq CE \int_t^T |Y_s - \bar{Y}_s|^2 ds. \end{aligned}$$

From Gronwall's lemma, we obtain $E|Y_t - \bar{Y}_t|^2 = 0, \forall t \in [0, T] - D$, and the result follows. \square

We come back to finish the Proof of Theorem 3.1:

Since Y is continuous, \bar{Y} is càdlàg, and D is countable, we get $Y_t = \bar{Y}_t$, P -a.s., $\forall t \in [0, T]$. Moreover, we can deduce that $M \equiv \bar{M}$. Hence, we get the result that the sequence $(Y_t^\varepsilon, \int_0^t Z_s^\varepsilon dB_s)$ converges in distribution to the process $(Y_t, \int_0^t Z_s d\bar{B}_s)$, and the proof of Theorem 3.1 is completed. \square

3.3. Examples

Example 3.1. Consider the case that \tilde{Q} is weakly irreducible with the state space $\mathcal{M} = \{1, \dots, m\}$ and $\nu = (\nu_1, \dots, \nu_m)$ is the quasi stationary distribution, then α^ε can be considered as a fast-varying noise process. As shown in the following, the noise is averaged out with respect to the quasi stationary distribution. In this case, the corresponding BSDE is

$$Y_t^\varepsilon = \xi + \int_t^T f(s, Y_s^\varepsilon, \alpha_s^\varepsilon) ds - \int_t^T Z_s^\varepsilon dB_s. \quad (12)$$

Under Assumption 2.1 and Assumption 3.1, from Theorem 3.1, as $\varepsilon \rightarrow 0$, the sequence of process $(Y_t^\varepsilon, \int_0^t Z_s^\varepsilon dB_s)$ converges in distribution to the process $(Y_t, \int_0^t Z_s d\bar{B}_s)$, where (Y, Z) is the unique solution to the following BSDE

$$Y_t = \xi + \int_t^T \sum_{i=1}^m \nu_i f(s, Y_s, i) ds - \int_t^T Z_s d\bar{B}_s. \quad (13)$$

It is noted that the generator of BSDE (13) depends on the quasi stationary distribution of the Markov chain. Thus we can adopt the distribution of a $\mathcal{F}_t^{\bar{B}}$ -adapted process Y , the solution of BSDE (13), as the asymptotic distribution for the solution of $\mathcal{F}_t^{\bar{B}} \vee \mathcal{F}_{t,T}^{\alpha^\varepsilon}$ -adapted process Y^ε .

In practical systems, the small parameter ε is just a fixed parameter and it separates different scales in the sense of order of magnitude in the generator. It does not need to tend to 0. We give a detailed example for interpretation.

Example 3.2. Suppose the generator of the continuous-time Markov chain affected BSDE (3) is $Q = \begin{pmatrix} -22 & 20 & 2 \\ 41 & -42 & 1 \\ 1 & 2 & -3 \end{pmatrix}$, and the corresponding state space is $\mathcal{M} = \{s_1, s_2, s_3\}$. It is obvious that the transition rate between s_1 and s_2 is larger than the transition rate between s_3 and other states, i.e., the jumps between s_1 and s_2 are more frequent than jumps between s_3 and other states. We can rewrite Q as following

$$Q = \frac{1}{0.05}\tilde{Q} + \hat{Q} = \frac{1}{0.05} \begin{pmatrix} -1 & 1 & 0 \\ 2 & -2 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} -2 & 0 & 2 \\ 1 & -2 & 1 \\ 1 & 2 & -3 \end{pmatrix}$$

It is noted that we choose suitable ε to guarantee that \tilde{Q} and \hat{Q} to be the generator with the same order of magnitude.

Now, we introduce the continuous-time ε -dependent singularly perturbed Markov chain $\alpha^\varepsilon = \{\alpha_t^\varepsilon; 0 \leq t \leq T\}$ which have the generator $Q^\varepsilon = \frac{1}{\varepsilon}\tilde{Q} + \hat{Q} = \frac{1}{\varepsilon} \begin{pmatrix} -1 & 1 & 0 \\ 2 & -2 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} -2 & 0 & 2 \\ 1 & -2 & 1 \\ 1 & 2 & -3 \end{pmatrix}$, and define the aggregated process

$$\bar{\alpha}^\varepsilon = \{\bar{\alpha}_t^\varepsilon; 0 \leq t \leq T\} = \begin{cases} 1, & \alpha_t^\varepsilon \in \{s_1, s_2\} \\ 2, & \alpha_t^\varepsilon \in \{s_3\} \end{cases}$$

Proposition 3.1 yields that $\bar{\alpha}^\varepsilon$ converges in distribution to a continuous-time Markov chain $\bar{\alpha}$ generated by $\bar{Q} = \begin{pmatrix} -\frac{5}{3} & \frac{5}{3} \\ 3 & -3 \end{pmatrix}$. By Theorem 3.1, we can adopt the probability distribution of the solution to the following BSDE

$$Y_t = \xi + \int_t^T \bar{f}(s, Y_s, \bar{\alpha}_s) ds - \int_t^T Z_s d\bar{B}_s$$

as an asymptotic probability distribution of the solution to the original BSDE. Here $\bar{f}(t, y, 1) = \frac{2}{3}f(t, y, s_1) + \frac{1}{3}f(t, y, s_2)$ and $\bar{f}(t, y, 2) = f(t, y, s_3)$.

Since the limit averaged Markov chain has two states and the original one has three states, we have reduced the complexity of the model. This advantage will be more clear when the state space of the original Markov chain is sufficiently larger.

4. Homogenization of One System of PDEs

As an application of our results in previous section, we show the homogenization of a sequence of semi-linear backward PDE with a singularly perturbed Markov chain. In this section, after showing the relation between BSDEs with Markov chain and one system of semi-linear PDE with Markov chain, we derive the homogenization property of backward PDE with a singularly perturbed Markov chain based on the weak convergence of the associated BSDE.

Here, we give some notations as follows: $C^k(R^p; R^q)$ is the space of functions of class C^k from R^p to R^q , $C_{l,b}^k(R^p; R^q)$ is the space of functions of class C^k whose partial derivatives of order less than or equal to k are bounded, and $C_p^k(R^p; R^q)$ is the space of functions of class C^k which, together with all their partial derivatives of order less than or equal to k , grow at most like a polynomial function of the variable x at infinity.

4.1. Relation between BSDEs with Markov chains and semi-linear PDEs systems with Markov chains

For $t \in [0, T]$, consider the following semi-linear backward PDE with a Markov chain:

$$u(t, x) = h(x) + \int_t^T (\mathcal{L}u(r, x) + f(r, x, u(r, x), (\nabla u \sigma)(r, x), \alpha_r)) dr, \quad (14)$$

here $u : [0, T] \times R^m \rightarrow R^k$, and $\mathcal{L}u = (Lu_1, \dots, Lu_k)'$, with $L = \frac{1}{2} \sum_{i,j=1}^m (\sigma \sigma')_{ij}$

$$(t, x) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^m b_i(t, x) \frac{\partial}{\partial x_i}.$$

Firstly, we make the following assumption:

Assumption 4.1. $b \in C_{l,b}^3(R^m; R^m)$, $\sigma \in C_{l,b}^3(R^m; R^{m \times d})$, $h \in C_p^3(R^m; R^k)$. For $f : [0, T] \times R^m \times R^k \times R^{k \times d} \times \mathcal{M} \rightarrow R^k$, $\forall s \in [0, T]$, $i \in \mathcal{M}$, $(x, y, z) \rightarrow f(s, x, y, z, i)$ is of class C^3 .

Moreover, $f(s, \cdot, 0, 0, i) \in C_p^3(R^m; R^k)$, and the first order partial derivatives in y and z are bounded on $[0, T] \times R^m \times R^k \times R^{k \times d} \times \mathcal{M}$, as well as their derivatives of order one and two with respect to x, y, z .

Definition 4.1. A classical solution of PDE (14) is a R^k -valued stochastic process $\{u(t, x); 0 \leq t \leq T, x \in R^m\}$ which is in $C^{0,2}([0, T] \times R^m; R^k)$ and satisfies that $u(t, x)$ is $\mathcal{F}_{t,T}$ -measurable.

$\forall t \in [0, T], x \in R^m$, we introduce the following FBSDE with a Markov chain on $[t, T]$:

$$X_s^{t,x} = x + \int_t^s b(X_r^{t,x}) dr + \int_t^s \sigma(X_r^{t,x}) dB_r, \quad (15)$$

$$Y_s^{t,x} = h(X_T^{t,x}) + \int_s^T f(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x}, \alpha_r) dr - \int_s^T Z_r^{t,x} dB_r. \quad (16)$$

The aim of this subsection is to show that, under above assumptions, the FBSDE (15)-(16) provides both a probabilistic representation and the unique classical solution for PDE (14).

For SDE (15), it is well known that under Assumption 4.1, it has a unique solution $\{X_s^{t,x}; t \leq s \leq T\}$ which has a version that is a.s. of class C^2 in x , the function and its derivatives are a.s. jointly continuous in (t, s, x) . Moreover,

$$\sup_{t \leq s \leq T} (|X_s^{t,x}| + |\nabla X_s^{t,x}| + |D^2 X_s^{t,x}|) \in \bigcap_{p \geq 1} L^p(R), \forall (t, x) \in [0, T] \times R^m$$

where $\nabla X_s^{t,x}$, $D^2 X_s^{t,x}$ denote respectively the matrix of first order and second order derivatives of $X_s^{t,x}$ with respect to x .

For BSDE (16), denote $\tilde{f}(s, y, z, i) = f(s, X_s^{t,x}, y, z, i)$, $\forall i \in \mathcal{M}$, we know that \tilde{f} satisfies Assumption 2.1 since f satisfies Assumption 4.1. So there exists a unique solution pair $\{(Y_s^{t,x}, Z_s^{t,x}); t \leq s \leq T\}$ to BSDE (16).

Define $X_s^{t,x} = X_{s \vee t}^{t,x}$, $Y_s^{t,x} = Y_{s \vee t}^{t,x}$, and $Z_s^{t,x} = 0$, for $s \leq t$. Then $(X, Y, Z) = (X_s^{t,x}, Y_s^{t,x}, Z_s^{t,x})$ is defined on $(s, t) \in [0, T]^2$.

Theorem 4.1. Under Assumption 4.1, let $\{u(t, x); 0 \leq t \leq T, x \in R^m\}$ be a classical solution of PDE (14). Suppose that there exists a constant C such that,

$$|u(t, x)| + |\partial_x u(t, x) \sigma(t, x)| \leq C(1 + |x|), \quad \forall (t, x) \in [0, T] \times R^m, \quad (17)$$

then $(Y_s^{t,x} = u(t, X_s^{t,x}), Z_s^{t,x} = \partial_x u(t, X_s^{t,x}) \sigma(t, X_s^{t,x}); t \leq s \leq T)$ is the unique solution of BSDE (16). Here $(X_s^{t,x}; t \leq s \leq T)$ is the solution to SDE (15).

Proof: $\forall t \leq s \leq T$, let $s = t_0 < t_1 < t_2 < \dots < t_n = T$, with Itô's formula and PDE (14), we get

$$\begin{aligned}
& Y_s^{t,x} - h(X_T^{t,x}) \\
&= u(s, x) - u(T, X_T^{t,x}) \\
&= \sum_{i=0}^{n-1} (u(t_i, X_{t_i}^{t,x}) - u(t_{i+1}, X_{t_{i+1}}^{t,x})) \\
&= \sum_{i=0}^{n-1} (u(t_i, X_{t_i}^{t,x}) - u(t_i, X_{t_{i+1}}^{t,x})) + \sum_{i=0}^{n-1} (u(t_i, X_{t_{i+1}}^{t,x}) - u(t_{i+1}, X_{t_{i+1}}^{t,x})) \\
&= \sum_{i=0}^{n-1} \left(- \int_{t_i}^{t_{i+1}} (\mathcal{L}u(t_i, X_s^{t,x}) ds - (\nabla u \sigma)(t_i, X_s^{t,x}) dB_s) \right. \\
&\quad \left. + \int_{t_i}^{t_{i+1}} (\mathcal{L}u(s, X_{t_{i+1}}^{t,x}) + f(s, X_{t_{i+1}}^{t,x}, u(s, X_{t_{i+1}}^{t,x}), (\nabla u \sigma)(s, X_{t_{i+1}}^{t,x}), \alpha_s)) ds \right).
\end{aligned}$$

(17) yields that

$$E \left(\sup_{t \leq s \leq T} |u(t, X_s^{t,x})|^2 + \int_t^T |\partial_x u \sigma(s, X_s^{t,x})|^2 ds \right) < \infty,$$

and the adaptability is obvious. The result is followed as $\Delta = \sup_{0 \leq i \leq n-1} |t_{i+1} - t_i| \rightarrow 0$. \square

Now we deduce the converse side of Theorem 4.1.

Theorem 4.2. Assume that for some $p > 2$, $E|\xi|^p + E \int_0^T |\tilde{f}(t, 0, 0, \alpha_t)|^p dt < \infty$, let b, σ, f, h, α satisfy Assumption 4.1, then the process $\{u(t, x) = Y_t^{t,x}; 0 \leq t \leq T, x \in R^m\}$ is the unique classical solution to PDE (14).

As preliminaries for the proof, we give two propositions about the regularity of the solution of BSDE (16) whose proofs are put in the Appendix.

Proposition 4.1. Under the assumption of Theorem 4.2, $\{Y_s^{t,x}; (s, t) \in [0, T]^2, x \in R^m\}$ has a version whose trajectories belong to $C^{0,0,2}([0, T]^2 \times R^m)$. Hence $\forall t \in [0, T]$, $x \rightarrow Y_t^{t,x}$ is of class C^2 a.s..

Proposition 4.2. Under the assumption of Theorem 4.2, $\{Z_s^{t,x}; (s, t) \in [0, T]^2, x \in R^m\}$ has an a.s. continuous version which is given by $Z_s^{t,x} =$

$\nabla Y_s^{t,x} (\nabla X_s^{t,x})^{-1} \sigma(X_s^{t,x})$. In particular, $Z_t^{t,x} = \nabla Y_t^{t,x} \sigma(x)$. Here $\left(\nabla Y_s^{t,x} = \frac{\partial Y_s^{t,x}}{\partial x}, \nabla Z_s^{t,x} = \frac{\partial Z_s^{t,x}}{\partial x} \right)$ is the unique solution of

$$\begin{aligned} \nabla Y_s^{t,x} = & h'(X_T^{t,x}) \nabla X_T^{t,x} + \int_s^T \left(f'_x(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x}, \alpha_r) \nabla X_r^{t,x} + f'_y(r, X_r^{t,x}, \right. \\ & \left. Y_r^{t,x}, Z_r^{t,x}, \alpha_r) \nabla Y_r^{t,x} + f'_z(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x}, \alpha_r) \nabla Z_r^{t,x} \right) dr - \int_s^T Z_r^{t,x} dB_r. \end{aligned}$$

Proof of Theorem 4.2: Let $t = t_0 < t_1 < \dots < t_n = T$, we have

$$\begin{aligned} & h(x) - u(t, x) \\ = & u(T, x) - u(t, x) \\ = & \sum_{i=0}^{n-1} (u(t_{i+1}, x) - u(t_i, x)) \\ = & \sum_{i=0}^{n-1} (u(t_{i+1}, x) - u(t_{i+1}, X_{t_{i+1}}^{t_i, x}) + u(t_{i+1}, X_{t_{i+1}}^{t_i, x}) - u(t_i, x)). \end{aligned}$$

Since $u(t_{i+1}, X_{t_{i+1}}^{t_i, x}) = Y_{t_{i+1}}^{t_{i+1}, X_{t_{i+1}}^{t_i, x}} = Y_{t_{i+1}}^{t_i, x}$, we obtain the following from BSDE (16)

$$\begin{aligned} & u(t_{i+1}, X_{t_{i+1}}^{t_i, x}) - u(t_i, x) \\ = & Y_{t_{i+1}}^{t_i, x} - Y_{t_i}^{t_i, x} \\ = & - \int_{t_i}^{t_{i+1}} f(r, X_r^{t_i, x}, Y_r^{t_i, x}, Z_r^{t_i, x}, \alpha_r) dr + \int_{t_i}^{t_{i+1}} Z_r^{t_i, x} dB_r. \end{aligned}$$

It is known that $u(t, \cdot) \in C^2(R^m)$ from Proposition 4.1. Then, with Itô's

formula, we get

$$\begin{aligned}
& h(x) - u(t, x) \\
&= \sum_{i=0}^{n-1} \left(\int_{t_i}^{t_{i+1}} \mathcal{L}u(t_{i+1}, X_r^{t_i, x}) dr - \int_{t_i}^{t_{i+1}} (\nabla u \sigma)(t_{i+1}, X_r^{t_i, x}) dB_r \right. \\
&\quad \left. - \int_{t_i}^{t_{i+1}} f(r, X_r^{t_i, x}, Y_r^{t_i, x}, Z_r^{t_i, x}, \alpha_r) dr + \int_{t_i}^{t_{i+1}} Z_r^{t_i, x} dB_r \right) \\
&= - \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} (\mathcal{L}u(t_{i+1}, X_r^{t_i, x}) + f(r, X_r^{t_i, x}, Y_r^{t_i, x}, Z_r^{t_i, x}, \alpha_r)) dr \\
&\quad + \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} (Z_r^{t_i, x} - (\nabla u \sigma)(t_{i+1}, X_r^{t_i, x})) dB_r.
\end{aligned}$$

From Proposition 4.1 and Proposition 4.2, letting $\Delta = \sup_{0 \leq i \leq n-1} |t_{i+1} - t_i| \rightarrow 0$, we have

$$u(t, x) = h(x) + \int_t^T (\mathcal{L}u(r, x) + f(r, x, u(r, x), (\nabla u \sigma)(r, x), \alpha_r)) dr,$$

here $u \in C^{0,2}([0, T] \times R^m; R^k)$. The uniqueness property is followed from Theorem 4.1 and the uniqueness of the solution of BSDE (16). \square

4.2. Homogenization of PDEs system with a singularly perturbed Markov chain

Now we can give the application of our theoretical result in previous section (Theorem 3.1): homogenization of PDEs system with a singularly perturbed Markov chain.

Consider the following sequence of semi-linear backward PDE with a singularly perturbed Markov chain, indexed by $\varepsilon > 0$, for $t \in [0, T]$, $x \in R^m$,

$$u^\varepsilon(t, x) = h(x) + \int_t^T (\mathcal{L}u^\varepsilon(r, x) + f(r, x, u^\varepsilon(r, x), \alpha_r^\varepsilon)) dr, \quad (18)$$

Here α^ε is the singularly perturbed Markov chain which is stated in subsection 3.1. We have the following homogenization result.

Theorem 4.3. *Under Assumption 3.1 and Assumption 4.1, PDE (18) has a classical solution $\{u^\varepsilon(t, x); 0 \leq t \leq T, x \in R^m\}$. As $\varepsilon \rightarrow 0$, the sequence*

of u^ε converges in distribution to a process u , where $u(t, x)$ is the classical solution of the following PDE with the limit averaged Markov chain $\bar{\alpha}$

$$u(t, x) = h(x) + \int_t^T (\mathcal{L}u(r, x) + \bar{f}(r, x, u(r, x), \bar{\alpha}_r)) dr, \quad 0 \leq t \leq T. \quad (19)$$

Here \bar{f} is the average of f defined as $\bar{f}(t, x, u, i) = \sum_{j=1}^{m_i} \nu_j^i f(t, x, u, s_{ij})$, for $i \in \bar{\mathcal{M}} = \{1, \dots, l\}$.

Proof. From Theorem 4.2, we know that $\{u^\varepsilon(t, x) = Y_t^{\varepsilon, t, x}; 0 \leq t \leq T, x \in R^m\}$ is the unique classical solution of PDE (18) where $\{Y_s^{\varepsilon, t, x}; t \leq s \leq T\}$ satisfies

$$Y_s^{\varepsilon, t, x} = h(X_T^{t, x}) + \int_s^T f(r, X_r^{t, x}, Y_r^{\varepsilon, t, x}, \alpha_r^\varepsilon) dr - \int_s^T Z_r^{\varepsilon, t, x} dB_r, \quad (20)$$

and $\{X_s^{t, x}; t \leq s \leq T\}$ satisfies SDE (15). $\forall (t, x) \in [0, T] \times R^m$, from Theorem 3.1, we obtain that $Y_t^{\varepsilon, t, x}$ converges in distribution to $Y_t^{t, x}$ as $\varepsilon \rightarrow 0$ where $\{Y_s^{t, x}; t \leq s \leq T\}$ satisfies

$$Y_s^{t, x} = h(X_T^{t, x}) + \int_s^T \bar{f}(r, X_r^{t, x}, Y_r^{t, x}, \bar{\alpha}_r) dr - \int_s^T Z_r^{t, x} d\bar{B}_r, \quad (21)$$

Again from Theorem 4.2, we know that $u(t, x) = Y_t^{t, x}$ is the unique classical solution to PDE (19), and the results are followed. \square

5. Conclusion

In this paper, stemmed from the adjoint equation for deriving the optimal control of stochastic LQ control problem with Markovian jumps, we study the solvability of one kind of BSDE with the generator depending on a Markov switching. Then, we consider the case that the Markov chain has a large state space. To reduce the complexity, we adopt a hierarchical approach and study the asymptotic property of BSDE with a singularly perturbed Markov chain. Also, as an application, we present the homogenization property of one system of PDE with a singularly perturbed Markov chain.

It is noted that in this paper, we only give the homogenization result of PDEs system with Markov chains when there exists classical solution under

smooth assumptions. In the successive work, we will study the Sobolev space weak solution for the related PDEs system and homogenization problem by virtue of BSDEs with Markov chain. Some applications of this kind of BSDEs in optimal control and mathematics financial problems would also be interesting to investigate in our future research.

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Appendix A. Proof of Proposition 4.1 and Proposition 4.2

The proof of Proposition 4.1 and Proposition 4.2 follow a classic approach as shown in [18, 19]. Here, we will give a sketch of the proof. Firstly, we present a higher order moment estimation to the solution of BSDE (3).

Corollary Appendix A.1. *Assume that for some $p > 2$, $E|\xi|^p + E \int_0^T |f(t, 0, 0, \alpha_t)|^p dt < \infty$, under Assumption 2.1, we have the following estimation for BSDE (3),*

$$E \left(\sup_{0 \leq s \leq t} |Y_s|^p + \left(\int_0^t Z_s^2 ds \right)^{\frac{p}{2}} \right) < \infty, \quad \forall 0 \leq t \leq T.$$

Proof. Applying Itô's formula to $|Y_t|^p$ from t to T , we can get

$$\begin{aligned} & |Y_t|^p + \frac{p(p-1)}{2} \int_t^T |Y_s|^{p-2} |Z_s|^2 ds \\ &= |\xi|^p + p \int_t^T |Y_s|^{p-2} Y_s f(s, Y_s, Z_s, \alpha_s) ds - p \int_t^T |Y_s|^{p-2} Y_s Z_s dB_s. \end{aligned}$$

By the same technique as that in Lemma 2.1 of Pardoux and Peng [18], we obtain that

$$\begin{aligned} & E|Y_t|^p + \frac{p(p-1)}{2} E \int_t^T |Y_s|^{p-2} |Z_s|^2 ds \\ & \leq E|\xi|^p + pE \int_t^T |Y_s|^{p-2} Y_s f(s, Y_s, Z_s, \alpha_s) ds. \end{aligned}$$

From Assumption 2.1, using Hölder and Young's inequalities, there exist $K > 0$ and C such that

$$\begin{aligned} & E|Y_t|^p + KE \int_t^T |Y_s|^{p-2} |Z_s|^2 ds \\ & \leq E|\xi|^p + CE \int_t^T (|Y_s|^p + |f(s, 0, 0, \alpha_s)|^p) ds. \end{aligned}$$

It follows from Gronwall's lemma that

$$\sup_{0 \leq t \leq T} E|Y_t|^p + E \int_0^T |Y_t|^{p-2} |Z_t|^2 dt < \infty.$$

Since

$$|Y_t|^p \leq |\xi|^p + p \int_t^T |Y_s|^{p-2} Y_s f(s, Y_s, Z_s, \alpha_s) ds - p \int_t^T |Y_s|^{p-2} Y_s Z_s dB_s,$$

Burkholder-Davis-Gundy inequality yields that $E(\sup_{0 \leq t \leq T} |Y_t|^p) < \infty$.

Now we prove $E(\int_0^t Z_s^2 ds)^{\frac{p}{2}} < \infty$. Since

$$\begin{aligned} \int_0^t Z_s dB_s &= Y_t - Y_0 + \int_0^t f(s, Y_s, Z_s, \alpha_s) ds, \\ \sup_{0 \leq t \leq T} \left| \int_0^t Z_s dB_s \right| &\leq 2 \sup_{0 \leq t \leq T} |Y_t| + \int_0^T |f(s, Y_s, Z_s, \alpha_s)| ds, \end{aligned}$$

the result is followed from Assumption 2.1 and Burkholder-Davis-Gundy inequality. \square

Lemma Appendix A.1. (Lemma 2.7 in [18]) For any $p > 2$, there exists a constant c_p such that for any $t, t' \in [0, T]$, $x, x' \in R^m$, $i \in \{1, \dots, d\}$, $h, h' \in R \setminus \{0\}$,

$$\begin{aligned} E\left(\sup_{0 \leq s \leq T} |X_s^{t,x}|^p\right) &\leq c_p(1 + |x|^p), \\ E\left(\sup_{0 \leq s \leq T} |X_s^{t,x} - X_s^{t',x'}|^p\right) &\leq c_p(1 + |x|^p)(|x - x'|^p + |t - t'|^{\frac{p}{2}}), \end{aligned}$$

$$E\left(\sup_{0 \leq s \leq T} |\Delta_h^i X_s^{t,x}|^p\right) \leq c_p,$$

$$E\left(\sup_{0 \leq s \leq T} |\Delta_h^i X_s^{t,x} - \Delta_{h'}^i X_s^{t',x'}|^p\right) \leq c_p(|x - x'|^p + |h - h'|^p + |t - t'|^{\frac{p}{2}}).$$

Here $\Delta_h^i g(x) = \frac{g(x + he_i) - g(x)}{h}$, $1 \leq i \leq d$, where e_i denotes the i th vector of an arbitrary orthonormal basis of R^m .

Proof of Proposition 4.1: Since

$$E\left(\sup_{0 \leq s \leq T} |X_s^{t,x}|^p\right) \leq c_p(1 + |x|^p),$$

from the proof of Corollary Appendix A.1, $\forall p > 2$, there exist C_p and q such that

$$E\left(\sup_{0 \leq s \leq t} |Y_s^{t,x}|^p + \left(\int_0^t |Z_s^{t,x}|^2 ds\right)^{\frac{p}{2}}\right) \leq C_p(1 + |x|^q).$$

Note that for $t \vee t' \leq s \leq T$

$$\begin{aligned} & Y_s^{t,x} - Y_s^{t',x'} \\ &= \left(\int_0^1 h'(X_T^{t,x} + \lambda(X_T^{t,x} - X_T^{t',x'})) d\lambda \right) (X_T^{t,x} - X_T^{t',x'}) \\ &+ \int_s^T \int_0^1 \left(f'_x(\Xi_{r,\lambda}^{t,x,t',x'}, \alpha_r)(X_r^{t,x} - X_r^{t',x'}) + f'_y(\Xi_{r,\lambda}^{t,x,t',x'}, \alpha_r)(Y_r^{t,x} - Y_r^{t',x'}) \right. \\ &\quad \left. + f'_z(\Xi_{r,\lambda}^{t,x,t',x'}, \alpha_r)(Z_r^{t,x} - Z_r^{t',x'}) \right) d\lambda dr - \int_s^T (Z_r^{t,x} - Z_r^{t',x'}) dB_r, \end{aligned}$$

where $\Xi_{r,\lambda}^{t,x,t',x'} = (r, X_r^{t',x'} + \lambda(X_r^{t,x} - X_r^{t',x'}), Y_r^{t',x'} + \lambda(Y_r^{t,x} - Y_r^{t',x'}), Z_r^{t',x'} + \lambda(Z_r^{t,x} - Z_r^{t',x'}))$. Since

$$E\left(\sup_{0 \leq s \leq T} |X_s^{t,x} - X_s^{t',x'}|^p\right) \leq c_p(1 + |x|^p)(|x - x'|^p + |t - t'|^{\frac{p}{2}}),$$

combing with the proof of Corollary Appendix A.1, we can deduce that $\forall p \geq 2$, there exist C_p and q such that

$$\begin{aligned} & E\left(\sup_{0 \leq s \leq T} |Y_s^{t,x} - Y_s^{t',x'}|^p + \left(\int_t^T |Z_s^{t,x} - Z_s^{t',x'}|^2 ds\right)^{\frac{p}{2}}\right) \\ & \leq C_p(1 + |x|^q)(|x - x'|^p + |t - t'|^{\frac{p}{2}}). \end{aligned}$$

Then using Kolmogorov's lemma, we know that $\{Y_s^{t,x}; (s, t) \in [0, T]^2, x \in R^m\}$ has an a.s. continuous version.

Next, we have

$$\begin{aligned} \Delta_h^i Y_s^{t,x} &= \int_0^1 h'(X_T^{t,x} + \lambda \Delta_h^i X_T^{t,x}) \Delta_h^i X_T^{t,x} d\lambda + \int_s^T \int_0^1 (f'_x(\Theta_{r,\lambda}^{t,x,h}, \alpha_r) \Delta_h^i X_r^{t,x} \\ &\quad + f'_y(\Theta_{r,\lambda}^{t,x,h}, \alpha_r) \Delta_h^i Y_r^{t,x} + f'_z(\Theta_{r,\lambda}^{t,x,h}, \alpha_r) \Delta_h^i Z_r^{t,x}) d\lambda dr - \int_s^T \Delta_h^i Z_r^{t,x} dB_r \end{aligned}$$

where $\Theta_{r,\lambda}^{t,x,h} = (r, X_r^{t,x} + \lambda h \Delta_h^i X_r^{t,x}, Y_r^{t,x} + \lambda h \Delta_h^i Y_r^{t,x}, Z_r^{t,x} + \lambda h \Delta_h^i Z_r^{t,x})$.

Since for each $p \geq 2$, there exists c_p such that

$$E\left(\sup_{0 \leq s \leq T} |\Delta_h^i X_s^{t,x}|^p\right) \leq c_p.$$

We can have the following estimation

$$E\left(\sup_{t \leq s \leq T} |\Delta_h^i Y_s^{t,x}|^p + \left(\int_t^T |\Delta_h^i Z_s^{t,x}|^2 ds\right)^{\frac{p}{2}}\right) \leq c_p(1 + |x|^q + |h|^q).$$

Then we consider

$$\begin{aligned} & \Delta_h^i Y_s^{t,x} - \Delta_{h'}^i Y_s^{t',x'} \\ &= \int_0^1 h'(X_T^{t,x} + \lambda h \Delta_h^i X_T^{t,x}) \Delta_h^i X_T^{t,x} d\lambda - \int_0^1 h'(X_T^{t',x'} + \lambda h \Delta_{h'}^i X_T^{t',x'}) \Delta_{h'}^i X_T^{t',x'} d\lambda \\ &+ \int_s^T \int_0^1 (f'_x(\Theta_{r,\lambda}^{t,x,h}, \alpha_r) \Delta_h^i X_r^{t,x} - f'_x(\Theta_{r,\lambda}^{t',x',h'}, \alpha_r) \Delta_{h'}^i X_r^{t',x'}) d\lambda dr \\ &+ \int_s^T \int_0^1 (f'_y(\Theta_{r,\lambda}^{t,x,h}, \alpha_r) \Delta_h^i Y_r^{t,x} - f'_y(\Theta_{r,\lambda}^{t',x',h'}, \alpha_r) \Delta_{h'}^i Y_r^{t',x'}) d\lambda dr \\ &+ \int_s^T \int_0^1 (f'_z(\Theta_{r,\lambda}^{t,x,h}, \alpha_r) \Delta_h^i Z_r^{t,x} - f'_z(\Theta_{r,\lambda}^{t',x',h'}, \alpha_r) \Delta_{h'}^i Z_r^{t',x'}) d\lambda dr \\ &- \int_s^T (\Delta_h^i Z_r^{t,x} - \Delta_{h'}^i Z_r^{t',x'}) dB_r. \end{aligned}$$

It is noted that

$$E\left(\sup_{0 \leq s \leq T} |\Delta_h^i X_s^{t,x} - \Delta_{h'}^i X_s^{t',x'}|^p\right) \leq c_p(|x - x'|^p + |h - h'|^p + |t - t'|^{\frac{p}{2}}).$$

$\forall i \in \mathcal{M}$, using similar arguments, we can show that

$$\begin{aligned} & E\left(\sup_{0 \leq s \leq T} |\Delta_h^i Y_s^{t,x} - \Delta_{h'}^i Y_s^{t',x'}|^p + \left(\int_{t \wedge t'}^T |\Delta_h^i Z_s^{t,x} - \Delta_{h'}^i Z_s^{t',x'}|^2 ds\right)^{\frac{p}{2}}\right) \\ & \leq c_p(1 + |x|^q + |x'|^q + |h|^q + |h'|^q) \times (|x - x'|^p + |h - h'|^p + |t - t'|^{\frac{p}{2}}). \end{aligned}$$

The existence of a continuous derivative of $Y_s^{t,x}$ with respect to x , and a mean-square derivative of $Z_s^{t,x}$ with respect to x follow from this estimation. And the existence of a continuous second derivative of $Y_s^{t,x}$ with respect to x

can be proved in a similar scheme. Using similar arguments as in the proof of Theorem 2.9 in [18], we can show that $\{Y_s^{t,x}; (s, t) \in [0, T]^2, x \in R^m\}$ has an a.s. continuous version. \square

Proof of Proposition 4.2: For any random variable F of the form $F = f(\varphi, B(h_1), \dots, B(h_n))$ with $f \in C_0^\infty(R^n)$, $\varphi \in L_{\mathcal{F}_T}^2$, $h_1, \dots, h_n \in L_{\mathcal{F}_t}^2(0, T; R^d)$ and $B(h_i) = \int_0^T h_i(t) dB_t$, where $\mathcal{F}_t = \mathcal{F}_{t,T}^\alpha \vee \mathcal{F}_t^B$, let $D_t F = \sum_{i=1}^n f'_i(B(h_1), \dots, B(h_n)) h_i(t)$, $0 \leq t \leq T$. For such F , we define its norm as

$$\|F\|_{1,2} = \left(E \left(F^2 + \int_0^T |D_t F|^2 dt \right) \right)^{\frac{1}{2}}.$$

Denote S as the set of random variables of the above form, we can define sobolev space: $D^{1,2} = \bar{S}^{\|\cdot\|_{1,2}}$. Using the same argument in Proposition 2.3 in [19], we can obtain the result. \square

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